M5007: Mathemagic with Cards

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1. The algebraic trick

The effect: The spectator shuffles a regular deck of 52 cards and hands it over to the magician. The magician deals the cards alternately face up and face down using the following rule: if the face-up card is a red card, it goes into a "red face-up" pile and the next immediate card goes into a "red face-down" pile; similarly, if the face-up card is a black card, it goes into a separate "black face-up" pile and the next immediate card goes into a "black face-down" pile (so there are 4 piles in total - the red face-up and face-down piles, and the black face-up and face-down piles). Note that all cards in the "red face-up" pile are actually red cards but all cards in the "red face-down" pile are not necessarily red cards, similarly all cards in the "black face-down" pile are not necessarily black cards.

After all 52 cards have been dealt out, the magician claims that he can make the number of red cards in the "red face-down" pile magically equal to the number of black cards in the "black face-down" pile. The magician asks the spectator to swap any number of cards between the "red face-down" pile and the "black face-down" pile, and while the spectator is swapping any number of cards between the two face-down piles, the magician casts his magic spell. Then the spectator counts the number of red cards in the "red face-down" pile and the number of black cards in the "black face-down" pile, and finds that they are indeed equal!

Performing the trick: This is a self-working trick, so just do it the way you saw it and it'll work, every single time!

The mathematics behind it: This trick can be understood by using simple algebra (refer to the table below while reading the explanation):

	Red pile		**Black pile**	
	No. of red cards	No. of black cards	No. of red cards	No. of black cards
Face-up pile	14	0	0	12
Face-down pile	x	14-x	12-x	x

Table showing the count of red and black cards in each of the 4 piles assuming that the "red face-up" pile has 14 cards (check that the trick works even if the "red face-up" pile has any other number of red cards).

- For the sake of explanation, let's say we end up with 14 red cards in the "red face-up" pile (the trick works for any number).
- Since we alternately dealt the 52 cards face up and face down, we know that the total number of face-up cards is 52/2 = 26, this implies that the number of black cards in the "black face-up" pile is 26-14 = 12.
- Now let's think about the cards in the face-down piles. We do not know how many red and black cards are present in either the "red face-down" pile or the "black face-down" pile. Let's denote the number of red cards in the "red face-down" pile by x (some unknown number).
- We know that there are 14 cards in the "red face-down" pile (since we started by saying that there are 14 cards in the "red face-up" pile, and we know that we dealt one card into the "red face-down" pile for every card that went into the "red face-up" pile). This implies that there are 14-x black cards in the "red face-down" pile.
- Since there are 26 black cards in total out of which 12 black cards are in the "black face-up" pile and 14-x black cards are in the "red face-down" pile, the remaining 26-12-(14-x) = x black cards must be in the "black face-down" pile! In other words, the number of black cards in the "black face-down" pile (whatever the exact number x might be) is exactly the same as the number of red cards in the "red face-down" pile! This is even before the spectator has swapped cards between the two piles.
- Swapping any number of cards between the red and the black face-down piles does not change the above analysis, so the count of red cards in the "red face-down" pile and black cards in the "black face-down" pile continue to remain equal although the actual count might change due to swapping (think and convince yourself why this must be true - start by thinking what happens when you swap exactly one card from each pile).

2. The parity trick

The effect: The magician asks the spectator to announce her favorite number (between 5 and 15[1]), and then asks her to randomly select that many "pairs" of cards from a regular deck of 52 cards for the trick. With the selected cards, the magician asks the spectator to do the following: cut the deck anywhere[2] and then turn over the top two cards as a pair[3]. The spectator is free to repeat this process as many times as she likes.

The magician then deals the cards alternately into two piles and then puts the two piles together. The magician then asks the spectator to count the number of face-up cards. The number of face-up cards turns out to be exactly the favorite number of the spectator.

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[1] The trick works for any number between 1 and 26, but the effect is the best when the number is not too low and not too high, so 5 to 15 is a good range.

[2] This means, split the deck anywhere into an upper set and a lower set, and then swap the two sets (so the upper set becomes the lower set and vice versa).

[3] In other words, pick the top two cards together and flip them over (so, for example, if the top two cards were initially face down, they would both become face up).

Performing the trick: This is another self-working trick but one important thing to note: while putting the two piles together, *invert one pile* before placing it on the other.

The mathematics behind it: This trick relies on two mathematical principles - parity and inversion of a set.

Simply put, *parity* is the state of being odd or even. In this trick, the parity of face-up cards is always even; in other words, face-up cards always exist in pairs[4]. This is because we start with all face-down cards, and the spectator always turns over the top two cards as a pair. This property ensures that when the magician deals the cards alternately into two piles, one face-up card of each pair goes into one pile and the other goes into the other pile; in other words, the face-up cards get equally split between the two piles.

Now when the magician puts the two piles together, he inverts one pile before placing it on the other. This is where *inversion of a set* comes into play. When you invert a set of n cards containing, let's say, x face-up cards and the remaining $(n-x)$ face-down cards, you end up with $(n-x)$ face-up cards and x face-down cards. When this inverted pile is placed on top of the other pile (which we know has x face-up cards since both piles had the same number of face-up cards to begin with), the total number of face-up cards in the deck is $(n-x) + x = n$. Thus, irrespective of how many cards were flipped by the spectators, the total number of face-up cards at the end of this trick is exactly equal to n which is the number of cards that go into each pile (which in turn is the favorite number that the spectator had announced at the start of the trick).

3. The information trick

The effect: The spectator shuffles a regular deck of 52 cards and randomly selects 8 cards for the trick. The magician asks the spectator to select one card in mind out of the 8 cards, and also to say a favorite number between 1 and 8. The magician then deals the cards into two piles, asks the spectator to point which pile her selected card is in, and combines the two piles together. The magician repeats this process two more times (so three rounds in total). At the end of it, the spectator's selected card magically appears at her favorite number-th position in the deck of 8 cards.

Performing the trick: To perform this trick, place the spectator's selected pile at the end of each round on top or bottom of the other pile according to the table on the next page.

The mathematics behind it: There are two parts to understanding this trick:

1. How does the magician know which is the selected card?

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[4] Sometimes there might be a lonely face-up card at the bottom and a lonely face-up card at the top, but if you imagine the cards are arranged cyclically, these two cards are part of the same pair.

2. How does the magician make the selected card appear at the spectator's favorite number-th position?

Favorite number	First round	Second round	Third round
1	Top	Top	Top
2	Bottom	Top	Top
3	Top	Bottom	Top
4	Bottom	Bottom	Top
5	Top	Top	Bottom
6	Bottom	Top	Bottom
7	Top	Bottom	Bottom
8	Bottom	Bottom	Bottom

How does the magician know which is the selected card?
Every time the spectator tells the magician which of the two piles her card is in, the magician is able to narrow down the set of possible cards by half. In other words, when the spectator tells the magician which pile her card is in at the end of the first round, the magician knows that the spectator has selected one of the 4 cards in that pile (that is, the magician has narrowed down the possible set of cards from 8 to 4). In the second round, these 4 cards get divided into 2 each in the two piles, so when the spectator tells which pile her card went into the second time, the magician has narrowed down the possibilities to 2 cards. In the third round, these 2 cards get divided into 1 each in the two piles, so when the spectator points out which pile her card went into, the magician exactly knows which was the selected card.

Alternative explanation: Another way of understanding why the above trick works is by using the concept of "information bits". In mathematics, information is actually quantifiable, and is measured in the unit of "bits" (1 or 0). To understand this better, let's say my dear friend Bob and I are playing a game. Bob says that his favorite number lies between 1 and 8, and I have to guess his favorite number. I am allowed to ask Bob only three Yes/No questions to guess his favorite number[5]. Note that a Yes or No from Bob gives me 1 bit of information (you can think of "Yes" corresponding to 1 and "No" to 0). My strategy would be to choose these questions such that each answer reduces the set of possible favorite numbers by half. My first question would be: "Is the number odd?" (I can also ask "Is the number even?"). If Bob answers with a "Yes", this means that his favorite number is one of 1, 3, 5 or 7 (otherwise, his favorite number is one of 2, 4, 6 or 8). Either ways, I have reduced my set of possible options by half. My second

[5] Have you ever played the "Twenty Questions" game? This is similar to that.

question would try to further reduce the set of possible numbers to two. For example, if Bob says his favorite number is odd, my second question would be “is the number 1 or 3?” (I could also ask if the number is 5 or 7 or any other possible pair). Let us say Bob confirms that his favorite number is indeed 1 or 3. My third question would finally help me nail down Bob’s favorite number if I ask him “Is the number 1?”.

Now let’s come back to the trick. The spectator is equally likely to pick any one of the 8 cards, so to know which of the 8 cards the spectator has actually selected, the magician needs to extract 3 bits of information. Each time the spectator tells the magician which pile her card went in to, she’s reducing the number of possibilities for her selected by a factor of 2, so every response is giving the magician 1 bit of information. After 3 rounds, the magician has 3 bits of information which is enough to know exactly which is the spectator’s selected card.

How does the magician make the selected card appear at the favorite number-th position?
After every round, the magician has 2 options - he can place the spectator’s selected pile either at the top or at the bottom of the other pile. Across all three rounds therefore, the magician has 2 x 2 x 2 = 8 options, and each option brings the selected card to a unique position between 1st and 8th in the deck. So by intelligently selecting where he places the spectator’s selected pile at the end of each of the 3 rounds, the magician can make the selected card appear at the spectator’s favorite number-th position.

As an example, let’s say the spectator’s favorite number is 7. The table above tells you that to make the card appear magically at the 7th position, you need to keep the spectator’s selected pile at the top (in the 1st round), bottom (in the 2nd round), bottom (in the 3rd round). Let’s now understand why this will work. If the spectator’s card were to appear at the 7th position after the 3rd round, then her selected pile in the 3rd round must be kept at the bottom of the other pile (if it were kept at the top, then her selected card can appear only at positions 1, 2, 3 or 4; if it’s kept at the bottom, then it can appear at positions 5, 6, 7 or 8). Now think about how this bottom pile came together - to make the spectator’s card appear at the third position in this pile of 4 cards, the spectator’s selected pile after the second round must be kept at the bottom (if it were kept at the top, it can only appear eventually at positions 5 or 6; if it’s kept at the top, it can appear eventually at positions 7 or 8). Now extend the same reasoning to the first round; the spectator’s selected pile in the first round must be kept at the top (if it were kept at the bottom, it’ll eventually appear at position 8; if it’s kept at the top, it’ll eventually appear at position 7).

An easy way of figuring out the top/bottom sequence without memorizing the above table is the following. Imagine you’re given the numbers 1 to 8 in sequence, and at every stage you can either pick the top half or the bottom half - what would you do to pick 7? You would pick the **bottom** half, then the **bottom** half again (of the bottom 4) and finally the **top** card (of the bottom 2). If you look at the table, the sequence you are looking for is exactly the reverse; that is, the sequence you follow is bottom (in the 3rd round), bottom (in the 2nd round) and top (in the 1st round) or (top, bottom, bottom).

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If you are excited to explore more of mathemagics, get started with the following amazing books:

Mathematics, Magic and Mystery

Martin Gardner
Courier Corporation, 1956

Mathematical Card Magic: Fifty-Two New Effects

Colm Mulcahy
CRC Press, 2013