CHAPTER 15

Special Relativity

From its publication in 1687 until 1905, Newtonian mechanics reigned supreme. It was applied to more and more systems, almost always with complete success. In those rare instances where Newtonian ideas appeared to fail, it was found that some complication had been overlooked, and, when this complication was included, Newton could again account for all the observations.[1] Newton's formulation was supplemented with new ideas (such as the notion of energy) and recast in different guises (by Lagrange and Hamilton), but the foundations seemed unshakeable. Then, toward the end of the nineteenth century, a few observations were made that seemed inconsistent with the classical, Newtonian, ideas. Heroic efforts were made to bring these observations into line with classical physics, but in 1905, Albert Einstein (1879–1955) published his first paper on the theory that we now call relativity, in which he showed that particles with speeds approaching the speed of light require a completely new form of mechanics, as I describe in this chapter. Even at slower speeds, Newtonian mechanics is only an approximation to the new "relativistic mechanics," but the difference is usually so small as to be undetectable. In particular, at the speeds usually encountered on earth, Newtonian mechanics is completely satisfactory, which explains why it is still a crucial and interesting part of physics (and justifies the other 15 chapters of this book).[2]

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[1] Perhaps the greatest such triumph for Newton was the prediction and discovery of the planet Neptune: Calculations of the orbit of Uranus (taking account of the other known planets, and based, of course, on Newtonian mechanics) disagreed with the observed position by some 1.5 minutes of arc. In 1846, it was shown independently by the English astronomer John Couch Adams (1819–1892) and the Frenchman Urbain Leverrier (1811–1877) that this discrepancy could be explained by the presence of a hitherto unnoticed planet outside the orbit of Uranus. Within a few months, the new planet, now called Neptune, was discovered by the German Johann Galle (1812–1910) at exactly its predicted position.

[2] In writing this chapter on relativity (particularly in the opening sections and the problems), it was sometimes difficult to resist borrowing ideas from the relativity chapters of *Modern Physics*, by Chris Zafiratos, Michael Dubson, and myself (second edition, Prentice Hall, 2003). I am grateful to Prentice Hall for giving me permission to do so.

15.1 Relativity

Let us first consider the significance of the name "relativity." A moment's thought should convince you that most physical measurements are made *relative* to a chosen reference system. That the position of a particle is $\mathbf{r} = (x, y, z)$ means that its position vector has components (x, y, z) *relative* to some chosen origin and a chosen set of axes. That an event occurs at time $t = 5$ s means that t is 5 seconds *relative* to a chosen origin of time, $t = 0$. If we measure the kinetic energy T of a car, it makes a big difference whether T is measured relative to a reference frame fixed on the road or to one fixed in the car. Almost all measurements require the specification of a reference frame, relative to which the measurement is to be made, and we can refer to this fact as the *relativity of measurements.*

The theory of relativity is the study of the consequences of the relativity of measurements. At first thought, this would seem unlikely to be a very interesting topic, but Einstein showed that a careful study of how measurements depend on the choice of coordinate system revolutionizes our whole conception of space and time, and requires a complete rethinking of Newtonian mechanics.

Einstein's relativity is really two theories. The first, called special relativity, is "special" in that it focuses primarily on unaccelerated frames of reference. The second, called general relativity, is "general" in that it includes accelerated reference frames. Einstein found that the study of accelerated frames leads naturally to a theory of gravitation, and general relativity turns out to be the relativistic theory of gravity. In practice, general relativity is required only in situations where its predictions differ appreciably from those of Newtonian gravity. These include the study of the intense gravity of black holes, of the large-scale universe, and of the effect of the earth's gravity on the extremely accurate time measurements needed for the global positioning system. In nuclear and particle physics, where we consider particles that move near the speed of light, but where gravity is usually completely negligible, special relativity is normally all that is needed. In this chapter, I shall treat only the special theory of relativity.[3]

15.2 Galilean Relativity

Many of the ideas of relativity are present in classical physics, and we have in fact met several in earlier chapters. Let us review these ideas and recast some of them in a form more suitable for our discussion of Einstein's relativity.

As we discussed in Chapter 1, Newton's laws hold in many different reference frames, namely, the so-called inertial frames, any one of which moves at constant velocity relative to any other. We can rephrase this to say that, in classical physics,

[3] To cover general relativity would require another book. Some good references are: R. Geroch, *General Relativity from A to B*, University of Chicago Press, 1978; I. R. Kenyon, *General Relativity*, Oxford University Press, 1990; B.F.A.Schutz, *A First Course in General Relativity*, Cambridge University Press, 1985; and James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*, Addison Wesley, 2003.

Newton's laws are **invariant** (that is, unchanged) as we transfer our attention from one inertial frame to another. The classical transformation from one frame to a second, moving at constant velocity relative to the first, is called the **Galilean transformation,** so a compact way to say the same result is that Newton's laws are invariant under the Galilean transformation. Let us first review this claim.

The Galilean Transformation

For simplicity, consider first two frames $\mathcal{S}$ and $\mathcal{S}'$ that are oriented the same way; that is, the x' axis is parallel to the x axis, y' parallel to y, and z' parallel to z. Suppose further that the velocity $\mathbf{V}$ of $\mathcal{S}'$ relative to $\mathcal{S}$ is along the x axis. It was a fundamental assumption of Newtonian mechanics that there is a single universal time t. Thus if the observers in $\mathcal{S}$ and $\mathcal{S}'$ agree to synchronize their clocks (and to use the same unit of time), then $t' = t$. Finally, we can choose our origins O and O' so that they coincide at the time $t = t' = 0$. This configuration is illustrated in Figure 15.1, where $\mathcal{S}$ is a frame fixed to the ground. (We'll assume that a frame fixed to the earth is inertial — that is, we'll ignore the slow rotation of the earth.) The frame $\mathcal{S}'$ is fixed in a train that is traveling with velocity $\mathbf{V}$ along the x axis.

Consider now some event, such as the explosion of a small firecracker. As measured by observers in $\mathcal{S}$ this occurs at position $\mathbf{r} = (x, y, z)$ and time t; as measured in $\mathcal{S}'$ it occurs at $\mathbf{r}' = (x', y', z')$ and time t'. Our first (and very simple) task is to establish the mathematical relation between the coordinates (x, y, z, t) and (x', y', z', t'). A moment's inspection of Figure 15.1 should convince you that $x' = x - Vt$, and that $y' = y$ and $z' = z$. By the classical assumption concerning time, $t' = t$, so the required relations are

$$\left.\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned}\right\} \tag{15.1}$$

These four equations are called the **Galilean transformation.** They give the coordinates (x', y', z', t') of any event as measured in $\mathcal{S}'$ in terms of the corresponding coordinates (x, y, z, t) of the same event as measured in $\mathcal{S}$. They are the mathematical expression of the classical ideas about space and time.

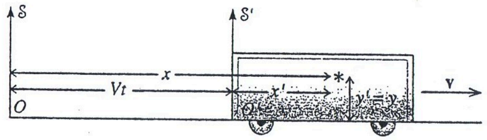

Figure 15.1 The frame $\mathcal{S}$ is fixed to the ground, while $\mathcal{S}'$ is fixed in a railroad car traveling with constant velocity $\mathbf{V}$ in the x direction. The two origins coincide, $O = O'$, at time $t = t' = 0$. The star indicates an event, such as a small explosion.

The Galilean transformation (15.1) relates the coordinates measured in two frames arranged with corresponding axes parallel and with relative velocity along the x axis, as shown in Figure 15.1 — an arrangement we can call the **standard configuration.** This is not, of course, the most general configuration. For example, if the relative velocity $\mathbf{V}$ is in an arbitrary direction, it is easy to see that (15.1) can be rewritten compactly as

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t \quad \text{and} \quad t' = t. \tag{15.2}$$

This is still not the most general form of the Galilean transformation, since we could rotate the axes, so that corresponding axes were no longer parallel, and we could displace the origins O or O' and the origins of time. However, (15.2) is general enough for our present purposes.

Using the Galilean transformation (15.2) we can immediately relate the velocities of an object, as measured in the two frames. If $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ is the velocity of the object as measured in $\mathcal{S}$ and $\mathbf{v}'(t)$ is likewise for $\mathcal{S}'$ then by differentiating (15.2) we find immediately that (remember that $\mathbf{V}$ is constant)

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}. \tag{15.3}$$

This is the **classical velocity-addition formula,** which asserts that, according to the ideas of classical physics, relative velocities add (or subtract) according to the normal rules of vector arithmetic.

Galilean Invariance of Newton's Laws

To prove the invariance of Newton's laws under the Galilean transformation, suppose that the second law holds in frame $\mathcal{S}$; that is, that $\mathbf{F} = m\mathbf{a}$, with all three variables measured in $\mathcal{S}$. Now it is an experimental fact (at least in the domain of classical mechanics) that measurements of the mass of any object give the same results in all inertial frames. Thus the mass m' measured in $\mathcal{S}'$ is the same as that measured in $\mathcal{S}$, and $m' = m$. The proof that the same is true for the net force depends, to some extent, on one's definition of force. If we take the view that forces are defined by the readings on spring balances, then it is clear that the force $\mathbf{F}'$ measured in $\mathcal{S}'$ is the same as that measured in $\mathcal{S}$, and $\mathbf{F}' = \mathbf{F}$. Finally, differentiating (15.3) with respect to time (and remembering that $\mathbf{V}$ is constant, by assumption) we see that $\mathbf{a}' = \mathbf{a}$. We have now proved that each of the variables $\mathbf{F}'$, m', and $\mathbf{a}'$ of frame $\mathcal{S}'$ is equal to the corresponding variable $\mathbf{F}$, m, and $\mathbf{a}$ of frame $\mathcal{S}'$. Therefore, if it is true that $\mathbf{F} = m\mathbf{a}$, it is also true that $\mathbf{F}' = m'\mathbf{a}'$. That is, Newton's second law is invariant under the Galilean transformation. I leave it as an exercise (Problem 15.1) to prove that the same is true of the first and third laws. The invariance of the laws of mechanics under the Galilean transformation was known to Galileo, who used it to argue that no experiment could tell whether the earth was "really" moving or "really" at rest, and hence that Kepler's sun-centered view of the solar system was just as reasonable as the traditional earth-centered view.

Galilean Relativity and the Speed of Light

While Newton's laws are invariant under the Galilean transformation, the same is not true of the laws of electromagnetism. Whether we write them in their compact form as Maxwell's four equations, or in their original form (as Coulomb's law, Faraday's law, and so on), they can be true in one inertial frame, but if they are, and *if the Galilean transformation were the correct relation between different inertial frames*, then they could not be true in any other inertial frame. By far the quickest way to verify this claim is to recall that Maxwell's equations imply that light (and, more generally, any electromagnetic wave) propagates through the vacuum in any direction with speed

$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}} = 3.00 \times 10^8 \text{ m/s}, \tag{15.4}$$

where ϵ_o and μ_o are the permittivity and permeability of the vacuum. Thus if Maxwell's equations hold in frame $\mathcal{S}$, then light must travel at the same speed c in any direction, as measured in $\mathcal{S}$. But now consider a second frame $\mathcal{S}'$, traveling at speed V along the x axis of $\mathcal{S}$, and imagine a beam of light traveling in the same direction. In $\mathcal{S}$ the light's speed is $v = c$. Therefore, in $\mathcal{S}'$ its speed is given by the classical velocity-addition formula (15.3) as

$$v' = c - V,$$

as shown on the left of Figure 15.2. Similarly, a beam of light traveling to the left will have speed $v = c$ in $\mathcal{S}$, but $v' = c + V$ in $\mathcal{S}'$. Depending on its direction, any beam of light will have speed v' (as measured in $\mathcal{S}'$) that varies anywhere between $c - V$ and $c + V$. Therefore, Maxwell's equations cannot hold in the inertial frame $\mathcal{S}'$.

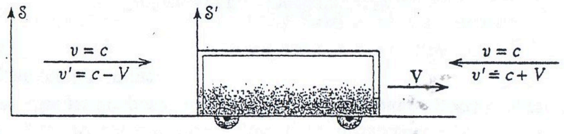

Figure 15.2 Two frames $\mathcal{S}$ and $\mathcal{S}'$ in the standard configuration with relative velocity $\mathbf{V}$. Two beams of light approach the car from opposite directions. If, as measured in $\mathcal{S}$, the light has speed c in either direction, then the classical velocity-addition formula implies that, as measured in $\mathcal{S}'$, it has speed $c - V$ traveling to the right, and $c + V$ traveling to the left.

If the Galilean transformation were the correct transformation between inertial frames, then although Newton's laws would hold in all inertial frames, there could only be one frame in which Maxwell's equations hold. This supposed unique frame, in which light would travel at the same speed in all directions, is sometimes called the ether frame.[4]

[4] The origin of the name is this: It was assumed that light must propagate through a medium, in much the same way that sound travels through the air. Since no one had ever detected this medium

The Michelson–Morley Experiment

The state of affairs just described, with the laws of mechanics valid in all inertial frames, but the laws of electromagnetism valid in a unique frame, was well understood toward the end of the nineteenth century. It was regarded by some (most notably Einstein) as unpleasing, and it was eventually shown by Einstein to be wrong. Nevertheless, it was logically consistent, and most physicists took for granted that there could be only one frame in which the speed of light had the same value c in all directions. Since the earth travels at a considerable speed in a continually changing direction around the sun, it seemed obvious that the earth must spend most of its time moving relative to the ether frame and hence that the speed of light as measured on earth should be different in different directions. The effect was expected to be very small. (The earth's orbital speed is $V \approx 3 \times 10^4$ m/s, large by terrestrial standards, but very small compared to $c = 3 \times 10^8$ m/s. Thus the fractional variation, between $c - V$ and $c + V$, was expected to be very small.) Nevertheless, in 1880, the American physicist Albert Michelson (1852–1931), later assisted by the chemist Edward Morley (1838–1923), devised an interferometer that should have easily detected the expected differences in the speed of light. To their surprise and dismay they found absolutely no variation.

Their experiments, and many different experiments with the same objective, have been repeated and have never found any reproducible evidence of variations in the speed of light relative to the earth. With hindsight, it is easy to draw the right conclusion: Contrary to all expectations, the speed of light is the same in all directions relative to an earth-based frame, even though the earth has different velocities at different times of year. In other words, it is not true that there is only one frame in which light has the same speed in all directions.

This conclusion is so surprising that is was not taken seriously for twenty years. Instead, several ingenious theories were advanced to explain the Michelson–Morley result while preserving the idea of a unique ether frame. For example, the so-called ether-drag theory held that the ether — the medium through which light was supposed to propagate — was dragged along with the earth, in much the same way the atmosphere *is* dragged along. This would imply that earth-bound observers are at rest relative to the ether and should measure the same speed of light in all directions. However, the ether-drag theory was incompatible with the phenomenon of stellar aberration.[5] None of these alternative theories was able to explain all of the observed facts (at least, not in a reasonable and economical way), and today almost all physicists accept that there is no unique ether frame and that the speed of light is a universal constant, with the same value in all directions in all inertial frames. The first person to

and since light could travel through seemingly empty space, the medium clearly had most unusual properties, and was named "ether" after the Greek for the stuff of the heavens. The "ether frame" was the frame in which the supposed ether was at rest.

[5] The ether-drag theory would require that light entering the earth's envelope of ether would be bent. This would contradict stellar abberation, in which the apparent position of any one star moves around a small circle as the earth moves around its circular orbit — in a way that makes clear that the light from the star travels in a straight line as it approaches the earth.

accept this surprising idea whole-heartedly was Einstein, as we now discuss. In particular, we shall see that the universality of the speed of light requires us to reject the Galilean transformation and the classical picture of space and time on which it was based. This, in turn, will require us to modify much of our Newtonian mechanics.

15.3 The Postulates of Special Relativity

The special theory of relativity is based on the acceptance of the universality of the speed of light, as suggested by the Michelson–Morley experiment.[6] Einstein proposed two postulates, or axioms, expressing his conviction that *all* the laws of physics should hold in all inertial frames, and from these postulates, he developed his special theory of relativity.

Before we discuss the postulates of relativity, it would be good to agree on what we mean by an inertial frame:

Definition of an Inertial Frame

An inertial frame is any reference frame (that is, a system of coordinates x, y, z and time t) in which all the laws of physics hold in their usual form.

Notice that I have not yet specified what "all the laws of physics" are. Following Einstein, we shall use the postulates of relativity to help us decide what the laws of physics could be. (As always, the ultimate test will be whether they agree with experiment.) It will turn out that one of the classical laws that carries over into relativity is the law of inertia, Newton's first law. Thus our newly defined inertial frames are in fact the familiar "unaccelerated" frames, where an object subject to no forces travels with constant velocity. As before, a frame fixed to the earth is (to a good approximation) inertial; a frame fixed to an accelerating rocket or a spinning turntable is not. The big difference between the inertial frames of relativity and those of classical mechanics is the mathematical relation between different frames. In relativity, we shall find that the classical Galilean transformation must be replaced by the so-called Lorentz transformation.

Notice also that I have specified that an inertial frame is one where the physical laws hold "in their usual form." As we saw in Chapter 9, one can sometimes modify physical laws so that they hold in noninertial frames as well. (For example, by introducing the centrifugal and Coriolis forces, we could use Newton's second law in a rotating frame.) It is to exclude such modifications that I added the qualifier "in their usual form."

[6] Whether Einstein actually knew about the Michelson–Morley result when he was formulating his theory is not clear. There is some evidence that he did, but it seems clear that his main motivation was the conviction that Maxwell's equations should hold in all inertial frames. Whether he knew or not affects neither Einstein's amazing accomplishment nor the importance of the Michelson–Morley result as beautifully clear evidence in favor of Einstein's assumptions.

The first postulate of relativity asserts the existence of many different inertial frames, traveling at constant velocity relative to one another:

First Postulate of Relativity

If $\mathcal{S}$ is an inertial frame and if a second frame $\mathcal{S}'$ moves with constant velocity relative to $\mathcal{S}$, then $\mathcal{S}'$ is also an inertial frame.

Another way to say this is that the laws of physics are invariant as we transfer our attention from one frame to a second one moving at constant velocity relative to the first. This is what we proved for the laws of mechanics, but we are now claiming it for *all* the laws of physics.

Another popular statement of the first postulate is that "there is no such thing as absolute motion." To understand this, consider two frames, $\mathcal{S}$ attached to the earth and $\mathcal{S}'$ attached to a rocket coasting at constant velocity relative to the earth. A natural question is whether there is any meaningful sense in which we could say that $\mathcal{S}$ is really at rest and $\mathcal{S}'$ is really moving (or vice versa). If the answer were "yes," then we could say that $\mathcal{S}$ is absolutely at rest and that anything moving relative to $\mathcal{S}$ is in absolute motion. However, this would contradict the first postulate of relativity: All of the laws observable by scientists in $\mathcal{S}$ are equally observable by scientists in $\mathcal{S}'$; any experiment that can be performed in $\mathcal{S}$ can equally be performed in $\mathcal{S}'$. Therefore, no experiment can show which frame is *really* moving. Relative to the earth, the rocket is moving; relative to the rocket, the earth is moving; and this is all we can say.

Yet another statement of the first postulate is that among all the inertial frames, there is no *preferred frame*. The laws of physics single out no one frame as being in any way more special than any other.

The second postulate specifies one of the laws that holds in all inertial frames:

Second Postulate of Relativity

The speed of light (in vacuum) has the same value c in every direction in all inertial frames.

This is, of course, the Michelson–Morley result.

Although the second postulate flies in the face of our everyday experience, it is by now a firmly established experimental fact. As we explore the consequences of Einstein's postulates we are going to encounter several surprising predictions, all of which seem to contradict our experience (for example, the phenomenon called time dilation, described in the next section). If you have difficulty accepting these predictions, there are two points to bear in mind: First, they are all logical consequences of the second postulate. Thus, once you have accepted the latter (surprising, but indisputably true), you *have* to accept all of its logical consequences, however counterintuitive they may seem. Second, all of these surprising phenomena (including the second postulate itself) have the subtle property that they become important only when objects travel with speeds comparable to the speed of light. In everyday life, with all speeds much less than c, these phenomena simply do not show up. In this sense, none of the surprising consequences of Einstein's postulates really conflict with our everyday experience.

15.4 The Relativity of Time; Time Dilation

Measurement of Time in a Single Frame

We are going to find that the second postulate forces us to abandon the classical notion of a single universal time. Instead, we shall find that the time of any one event, as measured in two different inertial frames, is in general different. This being the case, we need first to be quite clear what we mean by time, as measured in a single frame.

I shall take for granted that we have at our disposal lots of reliable tape measures and clocks. The clocks need not be identical, but they must have the property that, when brought together at the same point, at rest in the same inertial frame, they agree with one another. Let us now consider a single inertial frame $\mathcal{S}$, with origin O. We can station a chief observer at O with one of our clocks, and she can easily time any nearby event, such as a small explosion, since she will see it essentially instantaneously. To time an event farther away from the origin is harder, since light from the event has to travel to O before she can sense it. If she knew how far away the event occurred, then she could calculate how long the signal took to reach her (she knows that light travels at speed c) and subtract this from the time of arrival to give the time of the event. A simpler way to proceed (in principle anyway) is to employ a large number of helpers stationed at regular intervals throughout the region of interest and each with his own clock. The helpers can measure their distances from O, and we can check that their clocks are synchronized with the clock at O by having the chief observer send out a light signal at an agreed time (on her clock). Each helper can calculate the time taken by the signal to reach him and (allowing for this transit time) check that his clock agrees with the clock at O.

With enough helpers, stationed closely enough together, there will be a helper close enough to any event to time it essentially instantaneously. Once he has timed it, he can, at his leisure, inform everyone else of the result by any convenient means (such as a telephone). In this way, any event can be assigned a unique and well-defined time t as measured in the frame $\mathcal{S}$. In what follows, I shall assume that any inertial frame $\mathcal{S}$ comes with a set of rectangular axes $Oxyz$ and a team of helpers stationed at rest throughout $\mathcal{S}$ and equipped with synchronized clocks. This allows us to assign a position (x, y, z) and a time t to any event, as observed in the frame $\mathcal{S}$.

Time Dilation

Let us now compare measurements of times made by observers in two different inertial frames. Consider our familiar two frames, $\mathcal{S}$ anchored to the ground and $\mathcal{S}'$ traveling with a train in the x direction at speed V relative to $\mathcal{S}$. We now examine a **thought experiment** (or **gedanken experiment**, from the German) in which an observer on the train sets off a flashbulb on the floor of the train. The light travels to the roof, where

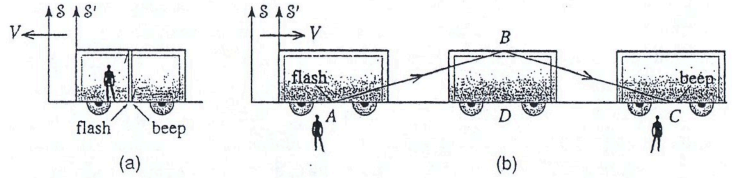

Figure 15.3 (a) The thought experiment as seen in frame $\mathcal{S}'$. The light travels straight up and down again, and the flash and beep occur at the same place. (b) As seen in $\mathcal{S}$, the flash and beep are separated by a distance $V\,\Delta t$. Notice that in $\mathcal{S}$ two observers are needed to time the two events, since they occur in different places.

it is reflected back and returns to its starting point, where it strikes a photocell and causes an audible "beep." We wish to compare the times, Δt and $\Delta t'$, as measured in the two frames, between the flash as the light leaves the floor and the beep as it returns.

As seen in the frame $\mathcal{S}'$, our experiment is shown in Figure 15.3(a). If the height of the train is h, then, as seen in $\mathcal{S}'$, the light travels a total distance $2h$ at speed c (second postulate) and so takes a time

$$\Delta t' = \frac{2h}{c}. \tag{15.5}$$

This is the time between the flash and the beep, as measured by an observer in $\mathcal{S}'$ (provided, of course, his clock is reliable).

As seen in $\mathcal{S}$, our experiment is shown in Figure 15.3(b). In particular, the same beam of light is seen to travel along the two sides AB and BC of a triangle. If Δt is the time between the flash and the beep (as measured in $\mathcal{S}$), the side AC has length $V\,\Delta t$. Thus the triangle ABD has sides[7] h, $V\Delta t/2$, and $c\Delta t/2$. (Notice that this is where we use the second postulate, that the speed of light is c in either frame.) Therefore,

$$(c\,\Delta t/2)^2 = h^2 + (V\Delta t/2)^2,$$

which we can solve to give

$$\Delta t = \frac{2h}{\sqrt{c^2 - V^2}} = \frac{2h}{c}\frac{1}{\sqrt{1-\beta^2}} \tag{15.6}$$

where I have introduced the useful abbreviation

$$\beta = \frac{V}{c}, \tag{15.7}$$

which is just the speed V measured in units of c.

[7] I take for granted that the height of the train is the same in either frame. We'll prove this shortly.

The striking thing about the two results (15.5) and (15.6) is that they are not equal. The time between the same two events (the flash and the beep) has different values as measured in the two different inertial frames. Specifically,

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}. \tag{15.8}$$

We derived this result for a thought experiment with a flash of light reflected back to its source by a mirror on the ceiling of the railroad car, but the conclusion applies to *any* two events that occur at the same place in the train. Suppose, for instance, an observer at rest in $\mathcal{S}'$ were to shout "Good" and a moment later "Grief." In principle, we could ignite a flashbulb at the "good," and arrange a mirror which would reflect the light back to arrive at the moment of "grief." Therefore, the relation (15.8) must apply to these two events, the "good" and the "grief." Since the timing of the two events cannot depend on whether we actually did the experiment with the light and the beeper, we conclude that the relation (15.8) must apply to *any* two events that occur at the same place in the frame $\mathcal{S}'$.

You should avoid thinking that the clocks in one of our frames are somehow running incorrectly — on the contrary, it was essential to our argument that all clocks, in both frames, were running correctly. Further, it makes no difference what particular kinds of clock we used, so the conclusion (15.8) applies to all (accurate) clocks. That is, *time itself*, as measured in the two frames, is different in accordance with (15.8). As we shall discuss shortly, this surprising conclusion has been verified repeatedly.

If the frame $\mathcal{S}'$ is actually at rest (relative to $\mathcal{S}$), then $V = 0$, so $\beta = 0$, and (15.8) reduces to $\Delta t' = \Delta t$. That is, there is no difference in the times unless $\mathcal{S}'$ is actually moving relative to $\mathcal{S}$. Moreover, at normal terrestrial speeds, $V \ll c$, so $\beta \ll 1$ and the denominator in (15.8) is very close to one. That is, at the speeds of our everyday experience, the two times are very nearly equal — so close that it would be almost impossible to detect any difference, as the following example shows.

EXAMPLE 15.1 Time Differences for a Jet Plane

Suppose that the pilot of a jet traveling at a steady $V = 300$ m/s arranges to set off a flashbulb at intervals of exactly one hour (as measured in his reference frame). If we arrange two observers on the ground to check this, what would they measure for the time Δt between two successive flashes? (Take the ground to be an inertial frame; that is, ignore effects of the earth's rotation.)

The required interval is given by (15.8) with $\Delta t' = 1$ hour and $\beta = V/c = 10^{-6}$. So

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}} = \frac{1\text{ h}}{\sqrt{1-10^{-12}}}$$

$$\approx 1\text{ h} \times (1 + \tfrac{1}{2} \times 10^{-12}) = 1\text{ h} + 1.8 \times 10^{-9}\text{ s}$$

where in going to the second line, I have used the binomial approximation.[8] In this experiment, the time difference is less than 2 nanoseconds (1 ns = 10^{-9} s). It is not hard to see why classical physicists had failed to detect such differences!

As we increase V, the difference between the times in (15.8) gets bigger, and if we let V approach c, we can make the difference as large as we please. For example, if $V = 0.99c$, then $\beta = 0.99$ and (15.8) gives $\Delta t' \approx 7\Delta t$. Speeds this high are routinely achieved by the accelerators at particle-physics labs, and the predicted time difference is precisely confirmed.

If we put $V = c$ (that is, $\beta = 1$) in (15.8), we would get the absurd result $\Delta t' = \Delta t/0$, and if we put $V > c$ (that is, $\beta > 1$), we would get an imaginary value for $\Delta t'$. These results suggest that V must always be less than c,

$$V < c,$$

a suggestion that proves correct and is one of the most profound results of relativity: The relative speed of two inertial frames can never equal or exceed c. That is, the speed of light, in addition to being the same in all inertial frames, is also the universal speed limit for the relative motion of any two inertial frames.

The factor $1/\sqrt{1-\beta^2}$ occurs so often in relativity, it usually given its own name, γ,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad (15.9)$$

It is useful to remember that this new factor always satisfies $\gamma \geq 1$, and as $\beta \to 1$ (that is, $V \to c$) $\gamma \to \infty$.

In terms of the parameter γ the result (15.8) can be written a little more compactly as

$$\Delta t = \gamma\, \Delta t' \geq \Delta t'. \qquad (15.10)$$

The asymmetry of this result (that $\Delta t'$ is never more than Δt) seems at first glance to violate the postulates of relativity, since it suggests a special role for the frame $\mathcal{S}'$— namely, that $\mathcal{S}'$ is the special frame in which the time interval is minimum. However, this is just as it should be, since in our thought experiment $\mathcal{S}'$ *is* special, because it is the frame where the two events in question (the flash and the beep) occur at the same place. (This asymmetry was implicit in Figure 15.3, which showed one observer measuring $\Delta t'$, but two measuring Δt.) To emphasize this asymmetry, the time $\Delta t'$ is often renamed Δt_o and (15.10) rewritten as

$$\Delta t = \gamma\, \Delta t_o \geq \Delta t_o \qquad (15.11)$$

[8] This is a nice example of a calculation where one almost *has to* use the binomial approximation, since most calculators cannot tell the difference between 1 and $1 - 10^{-12}$.

The subscript on Δt_o is to emphasize that Δt_o is the time elapsed on a clock at rest in the special frame where the two events in question occurred at the same place. This time is often called the **proper time** between the two events. In (15.11), Δt is the corresponding time measured in *any* frame and is always greater than or equal to the proper time Δt_o. For this reason, the effect implied by (15.11) is called **time dilation** and can be loosely stated by saying that *a moving clock is observed to run slow*. As measured by observers on the ground, a clock in the moving train is found to run slow.

Finally, I should emphasize the fundamental symmetry between any two inertial frames. We chose to do our thought experiment in a way that gave the frame $\mathcal{S}'$ a special role. (It was the frame in which the flash and beep occurred at the same place.) But we could have done the experiment the other way round, with the flashbulb, mirror and beeper at rest on the ground, and in this case, we would have found the opposite effect, that $\Delta t' = \gamma\, \Delta t$. The advantage of writing the time-dilation formula in the form (15.11) is that it avoids the problem of remembering which is frame $\mathcal{S}$, and which $\mathcal{S}'$; the subscript on Δt_o always flags the proper time — the time measured in the frame in which the two events were at the same place.

Evidence for Time Dilation

Time dilation was predicted in 1905 but was not experimentally verified until 1941, by B. Rossi and D. B. Hall.[9] The problem was, of course, to get a clock traveling sufficiently fast to show a measurable dilation. Rossi and Hall exploited the natural clocks that come with unstable subatomic particles, which decay (on average) after a definite time, characteristic of the particle. The lifetime of an unstable particle can be specified by its **half-life**, $t_{1/2}$, the time in which half of a large number of the particles will decay. The muon is an unstable particle that is created in the earth's upper atmosphere when cosmic ray particles (mostly protons and alpha particles) from outer space collide with atmospheric atoms. Many of these muons have speeds quite close to the speed of light, and they live long enough to find their way down to the earth's surface. The muon had been discovered in 1935 by Carl Anderson in his studies of cosmic rays. By 1941 its half-life was known to be about $t_{1/2} = 1.5\,\mu$s, meaning that half of a sample of muons *at rest* would decay in this time. If time dilation is correct, the half-life for a moving muon (as measured by earth-bound observers) should be larger by the factor γ as in (15.11). For example, if the muon had speed $0.8c$, then $\gamma = 1.67$, and the muon's half-life should be

$$t_{1/2}(\text{at speed } 0.8c) = 1.67 \times t_{1/2}(\text{at rest}) = 2.5\,\mu\text{s}.$$

Rossi and Hall were able to separate out cosmic-ray muons according to their speed and they could find their half-lives by measuring how many of them survived the journey through the atmosphere. Although their measurements had quite large experimental errors, they were nonetheless good enough to verify Einstein's prediction (15.11) and to exclude the classical assumption of a single universal time.

[9] B. Rossi and D. B. Hall, *Physical Review*, vol. 59, p. 223 (1941).

A test of time dilation using man-made clocks had to await the development of superaccurate atomic clocks. In 1971 four portable atomic clocks were synchronized with a reference clock at the U. S. Naval Observatory in Washington DC and then flown around the world in a jet plane and returned to the Naval Observatory. The observed discrepancy between the reference clock and the portable clocks was (273 ± 7) ns (averaged over the four clocks) in excellent agreement with the predicted value (275 ± 21) ns.[10]

Tests of time dilation — using both the natural clocks of unstable particles and man-made atomic clocks — have been repeated with ever-increasing precision, and there is now no doubt that the relativity of time, as embodied in (15.11), is true. Another important test that is carried out thousands of times every day is the Global Positioning System (GPS). This system, which is used by airplanes, ships, cars, and hikers to find their positions within a few meters, times the arrival of signals from 24 GPS overhead satellites at the observer's receiver and calculates the receiver's position from the known positions of the satellites. To find the position within a few meters requires an accuracy of a few nanoseconds, which requires that allowances be made for the relativistic differences between the times of the satellite and earth-bound reference frames. The success of the GPS is a daily tribute to the correctness of relativity.[11]

15.5 Length Contraction

The postulates of relativity have forced us to the conclusion that time is relative — the time between two given events is different when measured in different inertial frames — and, even more important, this conclusion is born out by experiment. This, in turn, implies that the length of an object is likewise dependent on the frame in which it is measured. To see this, we'll conduct a second thought experiment with the train of Figure 15.3, this time measuring its length. For an observer (let's call him Q) on the ground (frame $\mathcal{S}$) the simplest procedure is probably to measure the time Δt for the train to pass him and calculate the length as[12]

$$l = V\Delta t. \tag{15.12}$$

[10] See J. C. Hafele and R. E. Keating, *Science*, vol. 177, p. 166 (1972). Two trips were made, one going west and the other going east, both with satisfactory results. The numbers quoted here are for the more decisive westward trip. This experiment was actually a test of general, as well as special, relativity, since the predicted discrepancy has an appreciable contribution from gravitational effects.

[11] For a readable account of the large role of relativity in the GPS, see N. Ashby, *Physics Today*, May 2002, p. 41. As described there, there are important contributions from general, as well as special, relativity. Thus, the success of the GPS is a test of both theories.

[12] With so many of the familiar classical ideas being questioned, you are entitled to ask if it is legitimate to use the classical formula (15.12). However, this is just the definition of velocity (velocity = distance/time), and is certainly valid in any one reference frame (as long as we measure all quantities in this same frame).

To find the length l' of the train as measured in the train's rest frame, an observer on the train could simply use a long tape measure. However, for comparison with (15.12), it is convenient to use a different method. We can station two observers on the train, one at the front and another at the back, and have them record the times at which they pass the observer Q on the ground. The difference $\Delta t'$ between these two times is the time (as measured in frame $\mathcal{S}'$) for the train to pass observer Q, so the length of the train (again as measured in $\mathcal{S}'$) is just

$$l' = V\Delta t'. \tag{15.13}$$

Notice that we are making an important assumption here, that the speed of frame $\mathcal{S}$ relative to $\mathcal{S}'$ is the same as the speed V of $\mathcal{S}'$ relative to $\mathcal{S}$. (The relative velocities are in opposite directions, but their magnitudes are the same.) This is true in classical mechanics, and it is also true in relativity, where it follows from the two postulates. The details of the argument require some care, but the gist is this: Consider the transformation from frame $\mathcal{S}$ to $\mathcal{S}'$. We'll denote it by $(\mathcal{S} \to \mathcal{S}')$ temporarily. Suppose that, before making this transformation, we were to rotate our axes through 180° about the y (or z) axis, then make the transformation, and then rotate back again. The effect of the rotations is to reverse the direction of the x axis (and finally rotate it back again). The net effect of all three operations is precisely the transformation $(\mathcal{S}' \to \mathcal{S})$. Since the rotations certainly don't change any speeds, we've proved that the speed of $\mathcal{S}'$ relative to $\mathcal{S}$ is the same as that of $\mathcal{S}$ relative to $\mathcal{S}'$.

Comparing (15.12) with (15.13), we see that, since the times Δt and $\Delta t'$ are unequal, the same has to be true of the lengths l and l'. To quantify the difference, we must be careful to get the relation between Δt and $\Delta t'$ the right way around. These two times are the times (as measured in $\mathcal{S}$ and $\mathcal{S}'$) between two events: "front of train opposite observer Q" and "back of train opposite observer Q." These two events occur at the same place in frame $\mathcal{S}$, so Δt is the proper time, and $\Delta t' = \gamma\,\Delta t$. Inserting this into (15.13) and comparing with (15.12), we see that $l' = \gamma l$ or

$$l = \frac{l'}{\gamma} \le l'. \tag{15.14}$$

The length of the train as measured in $\mathcal{S}$ is less than that measured in $\mathcal{S}'$ (unless $V = 0$).

Like time dilation, the effect (15.14) is asymmetric, reflecting the asymmetry of the experiment. The frame $\mathcal{S}'$ is special, since it is the unique frame where the object being measured (the train) is at rest. [We could, of course, have done the experiment the other way round. If we had measured the length of a building that is at rest on the ground, then the roles of l and l' would have been reversed.] To avoid confusion as to which frame is which, it is common to rewrite (15.14) as

$$l = \frac{l_0}{\gamma} \le l_0 \tag{15.15}$$

where l_o denotes the length of an object measured in the object's **rest frame** (the frame in which the object is at rest), while l is the length in *any* frame. The length l_o is called the object's **proper length.** Since $l < l_o$ (if $V \neq 0$), this difference in lengths is called the **length contraction** (or the Lorentz contraction, or Lorentz–Fitzgerald contraction, after the two physicists — the Dutch Hendrik Lorentz, 1853–1928, and the Irish George Fitzgerald, 1851–1901 — who first suggested there must be some such effect.) The result can be loosely paraphrased by saying that *a moving body is observed to be contracted.*

Like time dilation, length contraction is a real effect, well established by experiment. Since the two effects are so intimately connected, any evidence for one can be taken as evidence for the other. In particular, the decay of a high-speed unstable particle, when viewed in the particle's rest frame, can be interpreted as clear evidence for length contraction. (See Problem 15.12.)

Lengths Perpendicular to the Relative Velocity

The length contraction just derived applies to lengths in the direction of the relative velocity, such as the length of a train in the direction of motion. It is easy to see that there can be no analogous contraction or expansion of lengths perpendicular to the motion, such as the height of the train. Suppose for example there were a contraction and imagine two observers Q standing at rest in $\mathcal{S}$ and Q' in $\mathcal{S}'$. Suppose further that Q and Q' are equally tall (when at rest) and that Q' is holding a knife exactly level with the top of his head. If there is a contraction, then as measured by Q, observer Q' will be shortened as he rushes past, and Q will be scalped, or worse, as the knife goes by. But, unlike our previous thought experiments, this experiment is completely symmetric between the two frames: There is just one observer in each frame, and the only difference is the direction of the relative velocities. Therefore, it must also be that, as seen by Q', it is Q who is contracted; so the knife misses Q, and Q is not scalped. The assumption of a contraction has led us to a contradiction and there can be no contraction. A similar argument excludes the possibility of expansion, and, in fact, the knife just scrapes past Q as seen in either frame. We conclude that lengths perpendicular to the relative motion are unchanged. The length-contraction formula (15.15) applies only to lengths parallel to the relative velocity.

15.6 The Lorentz Transformation

According to the classical notions of space and time, we saw that the mathematical relation between coordinates in two inertial frames $\mathcal{S}$ and $\mathcal{S}'$ is the Galilean transformation (15.1). In relativity, this cannot be the correct relation. (For example, time dilation contradicts the equation $t = t'$.) However, we can deduce the correct relation using an argument similar to the one that we used in connection with Figure 15.1 to derive the Galilean result. We imagine two frames, $\mathcal{S}$ attached to the ground and $\mathcal{S}'$ attached to a train moving with speed V relative to $\mathcal{S}$. We imagine, further, the explosion of a firecracker, which leaves a burn mark on the wall of the railroad car at a point

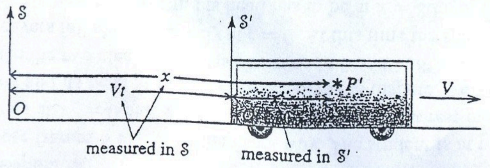

Figure 15.4 The coordinate x' is the horizontal distance, measured in $\mathcal{S}'$, between the origin O' and the burn mark at P'. The distances x and Vt are both measured in $\mathcal{S}$ at the time t (measured in $\mathcal{S}$) of the explosion.

P'. The coordinates of this explosion are (x, y, z, t) as measured by observers in $\mathcal{S}$ and (x', y', z', t') in $\mathcal{S}'$. Our object is to find formulas for x', y', z', and t' in terms of x, y, z, and t. The thought experiment is illustrated in Figure 15.4, which is just like Figure 15.1 except that we now know we must be very careful to identify the frames ($\mathcal{S}$ or $\mathcal{S}'$) relative to which the various distances are measured.

Since lengths perpendicular to the relative velocity are the same in both frames, we can immediately write

$$y' = y \quad \text{and} \quad z' = z \tag{15.16}$$

exactly as with the Galilean transformation. The coordinate x' is the horizontal distance between the origin O' and the burn mark at P', as measured in $\mathcal{S}'$. The same distance as measured in $\mathcal{S}$ is $x - Vt$, since x and Vt are the distances from O to P' and from O to O' at the instant t of the explosion (measured in $\mathcal{S}$). Therefore, by the length-contraction formula (15.15) (x' is the proper length here)

$$x - Vt = x'/\gamma$$

or

$$x' = \gamma(x - Vt). \tag{15.17}$$

This is the third of the four equations that we need. Notice that if $V \ll c$ then $\gamma \approx 1$ and (15.17) reduces to the Galilean relation $x' = x - Vt$.

Finally, to get an equation for t' we can use a simple trick. We could repeat the previous argument with the roles of $\mathcal{S}$ and $\mathcal{S}'$ exchanged. That is, we could let the explosion burn a mark at a point P on a wall fixed in $\mathcal{S}$. Arguing as before, we would get the result

$$x = \gamma(x' + Vt'). \tag{15.18}$$

(Notice that we could get this result directly from (15.17) by exchanging the primed and unprimed variables and replacing V by $-V$.) Substituting (15.17) into (15.18), we can eliminate x' and solve for t', to give (as you should check)

$$t' = \gamma(t - Vx/c^2). \tag{15.19}$$

This is the required equation for t'. When $V \ll c$, we can neglect the second term and $\gamma \approx 1$, so (15.19) reduces to the Galilean relation $t' = t$.

Equations assume S' moves in the +x direction → Lorentz Transformations

Collecting together the results (15.16), (15.17), and (15.19), we get the required four equations:

The Lorentz Transformation

$$\left.\begin{aligned} x' &= \gamma(x - Vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - Vx/c^2) \end{aligned}\right\} \quad (15.20)$$

These four equations are called the **Lorentz transformation** or the **Lorentz–Einstein transformation**, in honor of Lorentz, who first proposed them, and Einstein, who first interpreted them correctly. The Lorentz transformation gives the coordinates (x', y', z', t') of an event, as measured in $\mathcal{S}'$, in terms of its coordinates (x, y, z, t) as measured in $\mathcal{S}$. It is the correct relativistic version of the classical Galilean transformation (15.1).

If we wanted to know the coordinates (x, y, z, t) in terms of (x', y', z', t'), we could solve the four equations (15.20), but a simpler way is just to exchange primed and unprimed variables and replace V by $-V$. Either way, the result is the **inverse Lorentz transformation**

$$\left.\begin{aligned} x &= \gamma(x' + Vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + Vx'/c^2). \end{aligned}\right\} \quad (15.21)$$

The Lorentz transformation expresses all of the properties of space and time that follow from the postulates of relativity. Using it, one can calculate all of the kinematic relations between measurements made in different inertial frames. There are several examples of its use in the problems at the end of this chapter and here are a couple more.

EXAMPLE 15.2 Rederiving Length Contraction

Use the Lorentz transformation to rederive the length contraction formula (15.15). (Note that this will not give an alternative derivation of length contraction, since length contraction was used in deriving the Lorentz transformation. Rather we shall just get a consistency check.)

Consider our usual two frames, $\mathcal{S}$ fixed to the ground and $\mathcal{S}'$ fixed to a train traveling along the x axis with speed V relative to $\mathcal{S}$. We wish to compare the lengths of the train as measured in $\mathcal{S}$ and $\mathcal{S}'$. The measurement in $\mathcal{S}'$ is easy, since the train is at rest in this frame. An observer can, at his leisure, measure the x' coordinates x'_1 and x'_2 of the back and front of the train, and its length is just the difference $l' = x'_2 - x'_1$. This length is the proper length of the train, so

$$l_o = l' = x'_2 - x'_1. \quad (15.22)$$

The measurement in $\mathcal{S}$ is harder since the train is moving. We could, with enough care, station two observers Q_1 and Q_2 beside the track so that the back of the train passes Q_1 at the exact same instant $(t_1 = t_2)$ that the front passes Q_2. The length as measured in $\mathcal{S}$ is then just

$$l = x_2 - x_1.$$

Now, applying the Lorentz transformation (15.20) to the event "front of train passes Q_2" we get

$$x'_2 = \gamma(x_2 - Vt_2)$$

and, for the event "back of train passes Q_1,"

$$x'_1 = \gamma(x_1 - Vt_1).$$

Subtracting and remembering that $t_2 = t_1$, we find

$$l_o = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma l$$

or $l = l_o/\gamma$, which is the length contraction (15.15).

Our next example is one of the many seeming paradoxes of relativity.

EXAMPLE 15.3 A Relativistic Snake

A relativistic snake, of proper length 100 cm, is traveling across a table at $V = 0.6c$. To tease the snake, a physics student holds two cleavers 100 cm apart and plans to bounce them simultaneously on the table so that the left one lands just behind the snake's tail. The student reasons as follows: "The snake is moving with $\beta = 0.6$, so its length is contracted by the factor $\gamma = 5/4$ (check this) and its length measured in my frame is 80 cm. Therefore, the cleaver in my right hand bounces well ahead of the snake, which is unhurt." This scenario is shown in Figure 15.5. Meanwhile the snake reasons thus: "The cleavers are approaching me at $\beta = 0.6$, so the distance between them is contracted to 80 cm, and I shall certainly be cut to pieces when they fall." Use the Lorentz transformation to resolve this paradox.

Let us choose frames $\mathcal{S}$ and $\mathcal{S}'$ in the usual way. The student is at rest in $\mathcal{S}$, with the cleavers at $x_L = 0$ and $x_R = 100$ cm. The snake is at rest in $\mathcal{S}'$, with its tail at $x' = 0$ and its head at $x' = 100$. To resolve the dispute, we must find where and when the two cleavers fall, as observed in $\mathcal{S}$ and in $\mathcal{S}'$.

In $\mathcal{S}$ the cleavers fall simultaneously at $t = 0$. At this time the snake's tail is at $x = 0$. Since his length is 80 cm, his head has to be at $x = 80$ cm. [You can

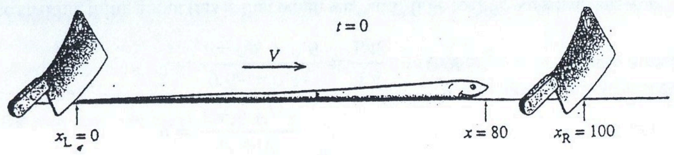

Figure 15.5 The snake paradox, as seen in the student's frame $\mathcal{S}$. The cleavers fall simultaneously at time $t = 0$.

check this, if you want, using the transformation equation $x' = \gamma(x - Vt)$; with $x = 80$ cm and $t = 0$, this gives the correct value $x' = 100$ cm.] As observed in $\mathcal{S}$, the experiment is as shown in Figure 15.5. The right cleaver falls comfortably ahead of the snake, the student is right, and the snake is unharmed.

What is wrong with the snake's reasoning? To answer this, we must examine the coordinates and times at which the two cleavers bounce, as observed in $\mathcal{S}'$. The left cleaver falls at $t_L = 0$ and $x_L = 0$. According to the Lorentz transformation (15.20), the coordinates of this event, as seen in $\mathcal{S}'$ are

$$t'_L = \gamma(t_L - Vx_L/c^2) = 0$$

and

$$x'_L = \gamma(x_L - Vt_L) = 0.$$

As expected, the left cleaver falls just behind the snake's tail, at time $t'_L = 0$, as shown in Figure 15.6(a).

So far there are no surprises. However, the right cleaver falls at $t_R = 0$ and $x_R = 100$ cm. Therefore, as seen in $\mathcal{S}'$, it falls at a time given by the Lorentz transformation as

$$t'_R = \gamma(t_R - Vx_R/c^2) = -2.5 \text{ ns}.$$

(Check the numbers yourself.) The crucial point is that, as seen in $\mathcal{S}'$, *the two cleavers do not fall at the same time*. Since the right cleaver falls *before* the left one, it does not necessarily hit the snake, even though they are only 80 cm apart

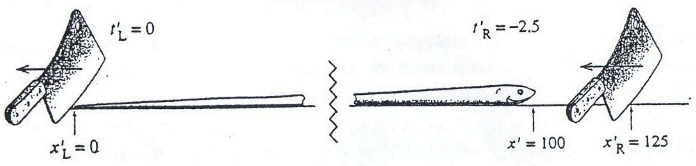

Figure 15.6 The snake paradox, as measured in the snake's frame $\mathcal{S}'$. The cleavers move to the left with speed V, and the right one falls 2.5 ns *before* the left one. Even though the cleavers are only 80 cm apart, this lets them land 125 cm apart.

(in this frame). In fact, the position at which the right cleaver falls is given by the Lorentz transformation as

$$x'_R = \gamma(x_R - Vt_R) = 125 \text{ cm}.$$

The right cleaver does indeed miss the snake!

The resolution of this paradox, and many similar paradoxes, is that two events that are simultaneous in one frame are not necessarily simultaneous in a different frame — an effect sometimes called the **relativity of simultaneity**. As soon as we recognize that the two cleavers fall at different times in the snake's frame, there is no longer any problem understanding how they can both contrive to miss the snake.

Stop Reading

15.7 The Relativistic Velocity-Addition Formula

As our next, and very important, application of the Lorentz transformation, let us use it to derive the relativistic velocity-addition formula. This formula is the answer to the following question: If an object — an electron, a baseball, a planet — is moving with velocity $\mathbf{v}$ relative to an inertial frame $\mathcal{S}$, how can we calculate its velocity $\mathbf{v}'$ relative to some other frame $\mathcal{S}'$? In classical physics, the answer to this question is the classical velocity-addition formula: If $\mathbf{V}$ denotes the velocity of $\mathcal{S}'$ relative to $\mathcal{S}$, then $\mathbf{v}' = \mathbf{v} - \mathbf{V}$. (Presumably, whoever named this formula wrote it as $\mathbf{v} = \mathbf{v}' + \mathbf{V}$.) For the special case that the axes of $\mathcal{S}$ and $\mathcal{S}'$ are parallel and $\mathbf{V}$ is in the x direction (our "standard" configuration), this becomes

$$v'_x = v_x - V, \quad v'_y = v_y, \quad \text{and} \quad v'_z = v_z. \tag{15.23}$$

Our task now is to find the corresponding relativistic result.

Consider a particle moving with position $\mathbf{r}(t)$ or $\mathbf{r}'(t')$, as seen in $\mathcal{S}$ or $\mathcal{S}'$. The definition of the velocity $\mathbf{v}$ is the derivative

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{15.24}$$

where $d\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is the infinitesimal displacement between the positions at times t_1 and $t_2 = t_1 + dt$. Now, we can write down the Lorentz transformation for (x_2, y_2, z_2, t_2) and (x_1, y_1, z_1, t_1), and taking differences, we find

$$dx' = \gamma(dx - Vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(dt - Vdx/c^2). \tag{15.25}$$

(Notice that $d\mathbf{r}$ and dt satisfy exactly the same transformation equations as $\mathbf{r}$ and t. This is because the Lorentz transformation turned out to be linear.) Using the definition (15.24), we can write down the components of $\mathbf{v}'$, and substituting (15.25) we find for v'_x

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - Vdt)}{\gamma(dt - Vdx/c^2)}$$